CRANBROOK SCHOOL

Year 12 MATHEMATICS

3 Unit (Second Paper), 4 Unit (First Paper)

Term 3 1999

Time: 2h

(GJB, MJB, LD, WMF, CGH, KMR, BES)

All questions may be attempted. All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Standard integrals are provided at the end of the paper.
Approved silent calculators may be used.
Begin each question in a new booklet.

1. (new booklet please)

a Solve $x^2 + x - 6 < 0$. **2**

b Differentiate $y = \log_e\left(\frac{x-1}{(x^2+1)}\right)$. **3**

c Find the exact value of $\int_0^{\frac{\pi}{2}} \cos 2x dx$. **3**

d Find a and b if $a + \sqrt{b} = (2 + 3\sqrt{2})^2$ **2**

e Sketch the graph of $y = \sqrt{3}x$. If the straight line $y = mx$ is added to the graph axes in such a way that the angle between $y = \sqrt{3}x$ and $y = mx$ is 60^0 find all possible values for m. **2**

2. (new booklet please)

a Show that $\log(\frac{3}{2}) + \log(\frac{4}{3}) + \log(\frac{5}{4}) + \ldots\ldots + \log(\frac{n}{n-1}) = \log(\frac{n}{2})$ **3**

b James invests \$1 500 at the end of each financial year in a superannuation fund for a period of 15 years. If the fund returns 5.5% pa over the period of investment, how much is in James' account at the end of 15 years? **4**

2 continued

c Triangle ABC has a right angle at A and angle C equal to 60^0. The side AC is 2 cm long. A circular arc is drawn with centre A, radius 2 cm cutting BC at D and AB at E.

[ED is an arc and DC a straight line.]

i Show that the area of the portion BED of the triangle is $\left(\sqrt{3}-\frac{\pi}{3}\right)$ cm^2. **2**

ii Find the area of the remaining portion ACDE of the triangle. **3**

3. (new booklet please)

a Using Induction, prove the following true for all positive integers, n

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$ **3**

b Write $5 \sin x - 3 \cos x$ in the form $r \sin(x-\alpha)$.

Hence solve the equation $5 \sin x - 3 \cos x = 2$ in the domain $-180° \le x \le 180°$ **3**

c When x metres from an origin on a straight line, the velocity v ms^{-1} of a particle is given by $v^2 = 9(3 + 2x - x^2)$

i Prove that the particle is in simple harmonic motion.

ii Find the centre of motion and the period. **6**

4. (new booklet please)

a A perfectly spherical balloon with radius r has a slow leak whereby the balloon remains spherical while slowly decreasing in size.

i If the gas is escaping at a rate of 2 m^3/min at a particular instant and the radius at that instant is 10 m, what is the rate of decrease in the surface area at that instant? **2**

ii Find the radius of the balloon at the instant when the rate of decrease of the surface area is numerically identical to the rate of decrease of the volume. **2**

b Consider the function $f(x) = 1 + \dfrac{2}{x-3}$ for $x > 3$.

i Give the equations of the horizontal and vertical asymptotes for $y = f(x)$. **2**

ii Find the inverse function $f^{-1}(x)$ **2**

iii State the domain of the inverse function. **1**

5. (new booklet please)

a Write $13+21+31+...+(n^2+n+1)$ using $\sum$ notation. 1

b The polynomial $P(x)=4x^3+3x^2+2$ has one real zero in the interval $-2<x<-1$. 6

i Show that this is the only real zero and sketch the graph of $P(x)$.

ii Using the approximation $x=-1\cdot 2$, find a better approximation using one application of Newton's method.

iii Explain what would happen if the approximation $x=-0\cdot 25$ had been used.

c Four men and four women are seated in a line. 2

i In how many different ways can the people be seated?

ii What is the probability that if one arrangement is chosen at random that the men and women are seated alternately?

d A biased coin has a probability of 0·6 of coming up Heads. If the coin is tossed 10 times what is the probability getting: 3

i 5 Heads?

ii 5 Heads, then 5 Tails?

iii At least two Heads, giving your answer to 4 significant figures?

6 (new booklet please)

a The graph below represents the velocity function of a particle moving in a straight line.

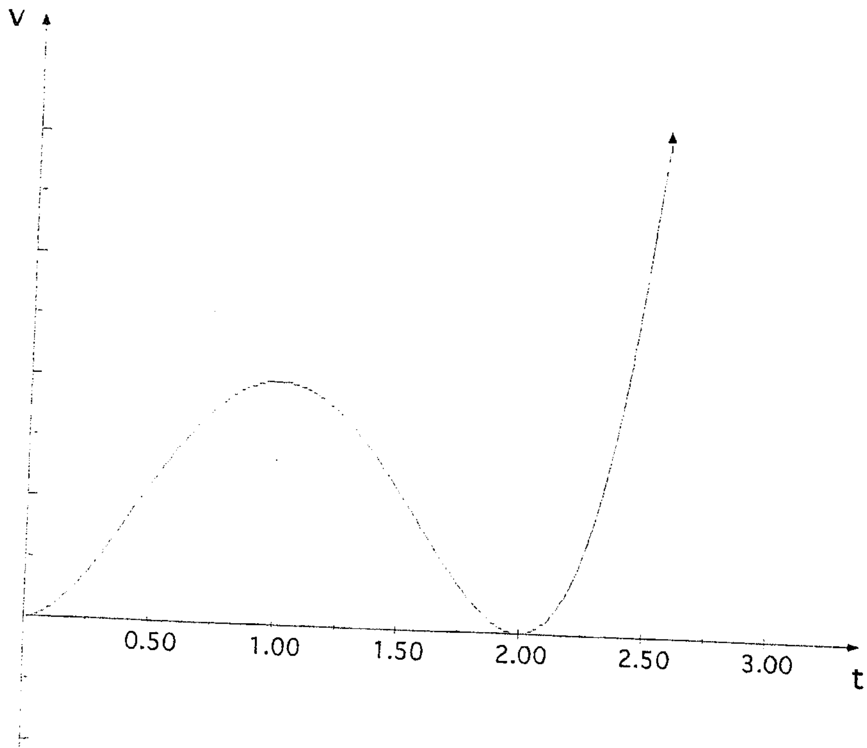
Copy the diagram into your answer booklet.

i State, with reason, whether the particle changes direction. 1

ii Indicate appropriately, on the diagram, the total distance travelled by the particle in the first two seconds. 1

iii On the same diagram, draw a curve that could be the displacement function of the particle. 2

6 **continued**



b A projectile is launched from the top of an 80 metre tower that stands on level ground. The particle's initial speed is 20 ms^{-1} and it's angle of projection is 60^o to the horizontal.

Assuming acceleration due to gravity is 10 ms^{-2}

i Derive the functions for the position of the projectile showing that $x = 10t$ and $y = 10\sqrt{3}t - 5t^2$ 2

ii Find the maximum height reached above ground level. 2

iii Find how long it takes for the projectile to hit the ground. 2

iv When the projectile is half way through its flight to the ground, calculate the direction of its flight. 2

7. (new booklet please)

a Given the polynomial $P(x) = x^3 - (k+1)x^2 + kx + 12$, 3

i Find the remainder when $P(x)$ is divided by $A(x) = x - 3$.

ii Find k if $P(x)$ is divisible by $A(x)$.

7 continued

b i Using the given substitution, evaluate $\int_0^3 x\sqrt{(x^2+1)^3}\,dx,\ u = x^2+1$ **4**

ii Find $\int \frac{x\,dx}{\sqrt{x^2+1}}$ **2**

c Find all real x such that $\frac{2}{|x-2|} > x+1$ **3**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} \qquad (n \neq -1;\ x \neq 0 \text{ if } n < 0)$$

$$\int \frac{1}{x}\,dx = \log_e x \qquad (x > 0)$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax \qquad (a \neq 0)$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax \qquad (a \neq 0)$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \qquad (a \neq 0)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} \qquad (a > 0,\ -a < x < a)$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log_e\left\{x+\sqrt{x^2-a^2}\right\} \qquad (|x| > |a|)$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log_e\left\{x+\sqrt{x^2+a^2}\right\}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} \qquad (a \neq 0)$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax \qquad (a \neq 0)$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax \qquad (a \neq 0)$$

I
3u

CRANBROOK SCHOOL
3/4u Trial
July 1999

Q1

Q1.(a) Solve $x^2 + x - 6 < 0 \Rightarrow (x+3)(x-2) < 0 \Rightarrow -3 < x < 2$ [2]

Q2.(b) diff $y = \log_e \dfrac{(x-1)}{(x^2+1)} \Rightarrow y = \log_e(x-1) - \log_e(x^2+1)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x-1} - \dfrac{2x}{x^2+1}$ [3]

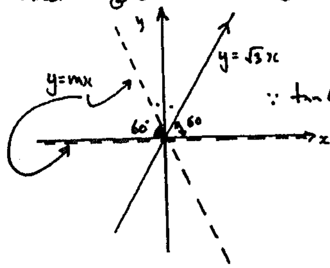
(c) Exact Value of $\int_0^{\frac{\pi}{2}} \cos 2x \, dx = \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left\{(\sin\pi) - (\sin 0)\right\}$ [3]

$= 0$

(d) Find a & b if $a + \sqrt{b} = (2+3\sqrt{2})^2 \Rightarrow a + \sqrt{b} = 22 + 12\sqrt{2}$

$\therefore a = 22$ & $b = 288$. [2]

(e)



$\because \tan 60° = \sqrt{3}$

$\therefore y = -\sqrt{3}x$ & $y = 0$ are at $60°$ to $y = \sqrt{3}x$

$\therefore m = -\sqrt{3}$ & 0 [2]

Q2(a) $\log\frac{3}{2} + \log\frac{4}{3} + \log\frac{5}{4} + \cdots + \log\frac{n}{n-1}$

LHS $= \log\left(\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\cdots\times\frac{n-1}{n-2}\times\frac{n}{n-1}\right) = \log\frac{n}{2} =$ RHS ✓✓✓ 3

(b) Y_1 1500×1.055

Y_2 $(1500 \times 1.055 + 1500)1.055 = 1500 \times 1.055^2 + 1500 \times 1.055$

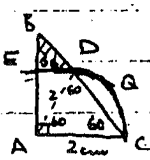
Y_3 $1500 \times 1.055^3 + 1500 \times 1.055^2 + 1500 \times 1.055$ ✓

Y_{15} $1500 \times 1.055^{15} + 1500 \times 1.055^{14} + \cdots + 1500 \times 1.055$

$S_{15} = 1500 \times 1.055\,(1.055^{14} + \cdots + 1)$ ✓

$= 1500 \times 1.055\left(\dfrac{1.055^{15} - 1}{0.055}\right) = \$35461.71$ ✓ ✓ 4

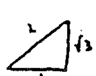
(c)



(i) Area $\triangle ABC = \frac{1}{2}.2.2.\tan 60° = 2\sqrt{3}$

Area quadrant ACDE: $\dfrac{\pi.2^2}{4} = \pi$ 3

$\tan 60° = \frac{x}{2}$ Area segment DQC $\frac{1}{2}r^2(\theta - \sin\theta)$



$= \frac{1}{2}.2^2\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$

$= 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$ ✓✓✓✓

Area BED $= 2\sqrt{3} - \pi + \frac{2\pi}{3} - \sqrt{3}$

$= \left(\sqrt{3} - \frac{\pi}{3}\right)$ cm^2

(ii) Area ACDE: $\pi - 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \pi - \frac{2\pi}{3} + \sqrt{3}$

$= \left(\sqrt{3} + \frac{\pi}{3}\right)$ cm^2 ✓

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QUESTION 3 - MARKED BY [illegible].

(a)
RTP $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\ldots+n^3$

Step 1:
Prove true for $n=1$

LHS $=1$ RHS $= \frac{1}{4}\times1^2(1+1)^2$
$= \frac{1}{4}\times1\times4$
$=1$
$=$ LHS

$\therefore$ true for $n=1$

Step 2:
Assuming true for $n=k$ ie
$1^3+2^3+3^3+\ldots+k^3 = \frac{1}{4}k^2(k+1)^2$,
prove true for $n=k+1$
ie. $1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

LHS $= 1^3+2^3+3^3+\ldots+k^3+(k+1)^3$
$= \frac{1}{4}k^2(k+1)^2+(k+1)^3$
$= \frac{1}{4}(k+1)^2(k^2+4(k+1))$
$= \frac{1}{4}(k+1)^2(k^2+4k+4)$
$= \frac{1}{4}(k+1)^2(k+2)^2$
$= \frac{1}{4}n^2(n+1)^2$ where $n=k+1$
$\therefore$ true for $n=k+1$

Step 3:
Since true for $n=1$ and true for $n=k+1$, (having assumed true for $n=k$) must be true for $n=1+1=2$, $n=2+1=3$ etc
$\therefore$ true for all +ve integers, n.

(4 marks)

(b) $5\sin x - 3\cos x$

(i) $r\sin(x-\alpha) = r\sin x\cos\alpha - r\cos x\sin\alpha$
$\therefore r\cos\alpha = 5$
$r\sin\alpha = 3$

$r = \sqrt{5^2+3^2}$ $\tan\alpha = \frac{3}{5}$
$= \sqrt{34}$ $\therefore \alpha = 30°58'$

$5\sin x - 3\cos x = \sqrt{34}\sin(x-30°58')$

$\therefore$ $5\sin x - 3\cos x = 2$
$\sqrt{34}\sin(x-30°58') = 2$
$\sin(x-30°58') = \frac{2}{\sqrt{34}}$
$x-30°58' = 20°4', 159°56'$ (w.A = $20°4'$)
$-200°4', -339°56'$
$\therefore x = 51°2', 190°54', -169°6', -30~~8~~58'$
(4 marks) out of domain, out of domain
$\therefore x = 51°2', -169°6'$

(c) $v^2 = 9(3+2x-x^2)$

(i) $\frac{d}{dx}(\frac{1}{2}v^2) = \frac{9}{2}(2-2x)$
$= 9(1-x)$
$\therefore \ddot{x} = -9(x-1)$ (2 marks)
$= -n^2(x-b)$ $n=3$, $b=1$
$\therefore$ motion is simple harmonic

(ii) Centre of motion: $x=1$ (2 marks)
period $= \frac{2\pi}{3}$

Question 4 Cranbrook HSC 1999 Trial

a(i) $\frac{dv}{dt} = -2$. Given $r = 10$.

To find $\frac{ds}{dt} = \frac{ds}{dr} \times \frac{dr}{dv} \times \frac{dv}{dt}$ ①.

$\therefore S = 4\pi r^2$ $\quad\quad V = \frac{4}{3} r^3$

$\frac{ds}{dr} = 8\pi r$.

$\frac{dv}{dr} = 4\pi r^2 \quad \therefore \frac{dr}{dv} = \frac{1}{4\pi r^2}$

From ① $\therefore \frac{ds}{dt} = 8\pi r \times \frac{1}{4\pi r^2} \times -2$.

$\frac{ds}{dt} = -\frac{4}{r}$

when $r = 10$. $\therefore \frac{ds}{dt} = -\frac{4}{10}$

$= -2/5 \ (0.4)$

$\therefore$ Rate of decrease in surface area is $0.4 m^2/s$.

(ii) Find r when $\frac{ds}{dt} = \frac{dv}{dt}$.

$\frac{ds}{dt} = \frac{-4}{r} \quad\quad \frac{dv}{dt} = -2$.

$\therefore \frac{-4}{r} = -2$

$\therefore r = 2m$.

Radius would be 2m.

b(i) Horizontal Asymptote $y = 1$
Vertical Asymptote $x = 3$.

bii $x = 1 + \frac{2}{y-3}$ will give inverse function of $f(x)$

$\therefore x - 1 = \frac{2}{y-3}$

$y - 3 = \frac{2}{x-1}$

(iii) Domain $x \neq 1$

3U Q5

IV / 3u

(a) $13+21+\dots+(n^2+n+1) = \sum_{r=3}^{n}(r^2+r+1)$ (1)

(b) (i) $P(x) = 4x^3+3x^2+2$

$P'(x) = 12x^2+6x$

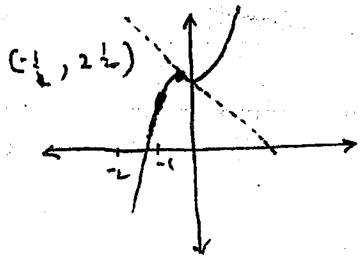
$= 6x(2x+1)$

$= 0$ if $x = 0, -\frac{1}{2}$

$P''(x) = 24x+6$

>0 if $x=0$ $\quad\therefore (0,2)$ min T.P.

<0 if $x=-\frac{1}{2}$ $\quad\therefore (-\frac{1}{2}, 2\frac{1}{4})$ max TP



(2)

Because local min^m is >0 as shown on graph, (1) there is only one ZE

(ii) $x_2 = x_1 - \dfrac{P(x_1)}{P'(x_1)}$

$= -1.2 - \dfrac{-.592}{10.08}$ (1)

$= -1.1413\dots$ (1)

(iii) Using $x_1 = -.25$ gives a tangent that slopes away from the zero (eg dotted line on graph) (1)

(c) (i) $8!$ (40320) (1)

(ii) $\dfrac{4!\times 4!\times 2}{8!}$ (1) $(\frac{1}{35})$ [4! ways of arranging each group ; 2! ways of deciding whether L.H. person is M or F]

(d) (i) $\binom{10}{5}(.6)^5(.4)^5$ (1) $(.2006\dots)$ [$(.6)^5$ term in expansion of $(.6+.4)^{10}$]

(ii) $.6^5\times.4^5$ (1) $(.000796\dots)$

(iii) $1-\left[\binom{10}{0}.4^{10}+\binom{10}{1}.6\times.4^9 + \binom{10}{2}.6^2\times.4^8\right]$ (1) (complement of 0H or 1H

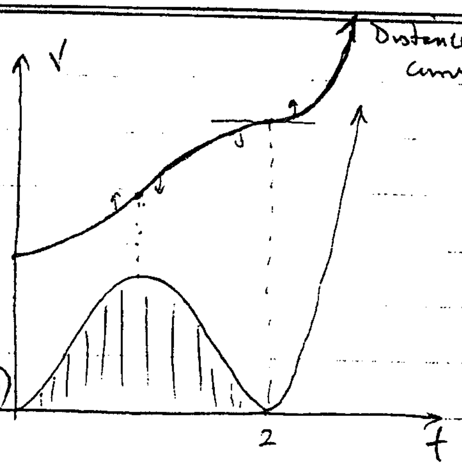
$= 1-.001677\dots$

$= \underline{\underline{.9983}}$

1999 3U HSC TRIAL – QUESTION 6

6. (a)

(this curve can be raised or lowered or stretched vertically with x on the vertical axis)

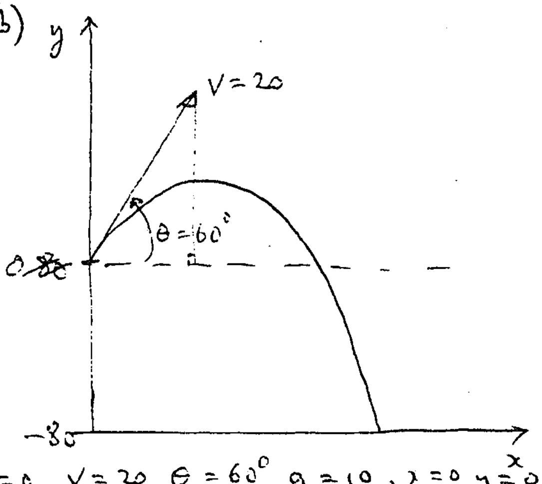


(i) The velocity never becomes negative $\therefore$ the particle does not change direction. [1]

(ii) shaded on diagram. [1]

(iii) drawn on diagram. [2]

(b)



$t=0 \quad V=20 \quad \theta=60^\circ \quad g=10, \; x=0, y=0,$

$\frac{dx}{dt} = 20\cos 60^\circ \qquad \frac{dy}{dt} = 20\sin 60^\circ$

$= 10 \qquad\qquad = 10\sqrt{3}$

i) $\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = -g = -10$

$\therefore \frac{dx}{dt} = c_1 \qquad \therefore \frac{dy}{dt} = -10t + c_2$

but from boundary conditions above $c_1 = 10$

and $c_2 = 10\sqrt{3}$.

$\therefore \boxed{\frac{dx}{dt} = 10}$ and $\boxed{\frac{dy}{dt} = 10\sqrt{3} - 10t}$

$x = 10t + c_3 \qquad y = 10\sqrt{3}t - \frac{10t^2}{2} + c_4$

but when $t=0$ $x=0, y=0 \therefore c_3 = c_4 = 0$

$\therefore \boxed{x = 10t}$ and $\boxed{y = 10\sqrt{3}t - 5t^2}$ [2]

(ii) Maximum height reached at half time of flight (to position level with projection)

ie when $y=0$ $\quad 10\sqrt{3}t - 5t^2 = 0$

$\therefore 5t(2\sqrt{3} - t) = 0$

$\therefore t = 0$ or $2\sqrt{3}$.

so half time of flight is $\sqrt{3}$ seconds

& Maximum height is $y = 10\sqrt{3}\times\sqrt{3} - 5\times\sqrt{3}^2$

$= 30 - 15$

$= 15$ metres

above the ground ie 95 metres [2]

(iii) Projectile hits ground when $y = -80$

ie $10\sqrt{3}t - 5t^2 = -80$.

$\therefore t^2 - 2\sqrt{3}t - 16 = 0$.

$t = \frac{2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4\times1\times-16}}{2}$

$= \frac{2\sqrt{3} \pm \sqrt{12 + 64}}{2}$

$= \frac{2\sqrt{3} \pm \sqrt{76}}{2}$

Since $t > 0$

$t = \frac{2\sqrt{3} + 2\sqrt{19}}{2}$ [2]

$= 6.0909\ldots$ seconds.

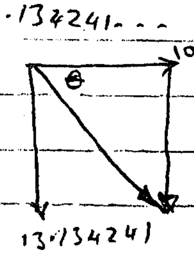
(iv) When $t = 3.04545\ldots$

$\frac{dx}{dt} = 10$ and $\frac{dy}{dt} = 10\sqrt{3} - 10\times3.045\ldots$

$= -13.134241\ldots$

$\tan\theta = \frac{13.134241\ldots}{10}$

$\therefore \theta = 52.7^\circ$



$\therefore$ the direction is 52.7° downwards from the horizontal.

7. (a) $P(x) = x^3 - (k+1)x^2 + kx + 12$

(i) $P(3) = 27 - 9(k+1) + 3k + 12$

$= 27 - 9k - 9 + 3k + 12 = \underline{30 - 6k}$ 2

(ii) $P(x)$ divisible by $(x-3)$ if $30 - 6k = 0$

i.e. $\underline{k = 5}$ 1

(b) (i) $\displaystyle\int_0^3 x\sqrt{(x^2+1)^3}\,dx$

Set $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{dx}{du} = \frac{1}{2x}$ 1

If $x = 0,\ u = 1$

$x = 3,\ u = 10$

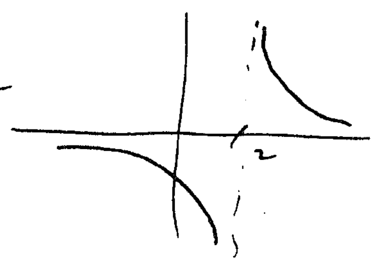
$= \displaystyle\int_1^{10} \cancel{x}\, u^{3/2} \cdot \frac{1}{2\cancel{x}}\,du$

$= \frac{1}{2}\left[\frac{2}{5}u^{5/2}\right]_1^{10}$ 1 $= \frac{1}{5}\left[10^{5/2} - 1^{5/2}\right] = \underline{\dfrac{100\sqrt{10} - 1}{5}}$ 1

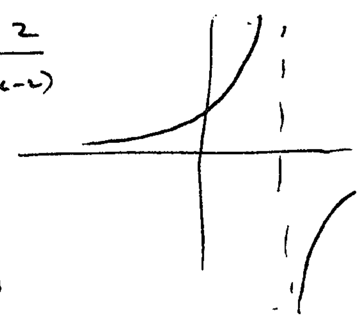
(ii) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{2x}{(x^2+1)^{1/2}}\,dx$

$= \frac{1}{2} \cdot 2(x^2+1)^{1/2} + c = \underline{\sqrt{x^2+1} + c}$ 2

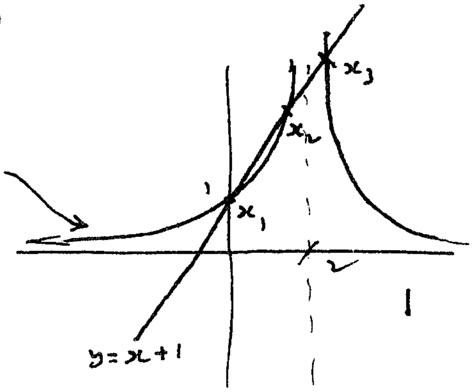
(c) Graphs $y = \dfrac{2}{x-2}$



$y = \dfrac{2}{-(x-2)}$



$\therefore\ y = \dfrac{2}{|x-2|}$



x coordinates of pts of intersection are x_1, x_2, x_3

1

x_1, x_2: $\dfrac{2}{2-x} = x+1$

$\therefore\ 2 = 2x + 2 - x^2 - x$

$x^2 - x = 0$

$x(x-1) = 0$

$\underline{x = 0, 1}$ 1

$x = 0,\ x = 1$

x_3: $\dfrac{2}{x-2} = x+1$

$2 = x^2 - x - 2$

$x^2 - x - 4 = 0$ 1

$x_3 = \dfrac{1+\sqrt{17}}{2}$

$\therefore$ From graph

$\dfrac{2}{|x-2|} > x+1$

if $\underline{x < 0,\ 1 < x < \dfrac{1+\sqrt{17}}{2}}$ 1